Journal of Computational and Applied Mechanics, Vol. 19, No. 1, (2024), pp. 3–15
DOI: 10.32973/jcam.2024.001

INVESTIGATION OF THE STRESS STATE IN A THIN ELASTIC DISC

István Ecsedi
Institute of Applied Mechanics, University of Miskolc
Miskolc-Egyetemváros, H-3515 Miskolc, Hungary
mechecs@uni-miskolc.hu

Ákos József Lengyel
Institute of Applied Mechanics, University of Miskolc
Miskolc-Egyetemváros, H-3515 Miskolc, Hungary
mechlen@uni-miskolc.hu

[Received: July 6, 2023; Accepted: March 25, 2024]

Abstract. This paper deals with the stress state in a thin elastic disc which is loaded by a uniform radial load on its outer curved boundary surface. Two solutions are presented. The first is an elastic solution based on the governing equation of the plane stress state. The second is a strength of material solution. The results obtained from the plane stress model are compared to those obtained from the strength of material solution.

Mathematical Subject Classification: 74A10, 74B05, 74G10, 74K20
Keywords: Thin elastic disc, plane stress, strength of materials

1. Formulation of the boundary value problem

Figure 1 shows the thin elastic disc which is loaded by uniform radial load on its outer curved boundary part. The plane domain of the middle section of the elastic disc is denoted by A and the boundary curve of A is $\partial A = \partial A_1 \cup \partial A_2 \cup \partial A_3 \cup \partial A_4$. The formulation of the boundary value problem is presented in the $Or\varphi$ cylindrical coordinate system. It is evident that

$$A = \{(r,\varphi)\,|a \le r \le b,\ 0 \le \varphi \le \pi\}\,, \tag{1.1}$$

$$\partial A_1 = \{(r,\varphi)\,|a \le r \le b,\ \varphi = 0\}\,, \tag{1.2}$$

$$\partial A_2 = \{(r,\varphi)\,|r = b,\ 0 \le \varphi \le \pi\}\,. \tag{1.3}$$

$$\partial A_3 = \{(r,\varphi)\,|a \le r \le b,\ \varphi = \pi\}\,, \tag{1.4}$$

$$\partial A_4 = \{(r,\varphi)\,|r = a,\ 0 \le \varphi \le \pi\}\,. \tag{1.5}$$

The displacement vector $\mathbf{t} = \mathbf{t}(r,\varphi)$ can be represented as

$$\mathbf{t}\,(r,\varphi) = u\,(r,\varphi)\,\mathbf{e}_r + v\,(r,\varphi)\,\mathbf{e}_\varphi, \tag{1.6}$$

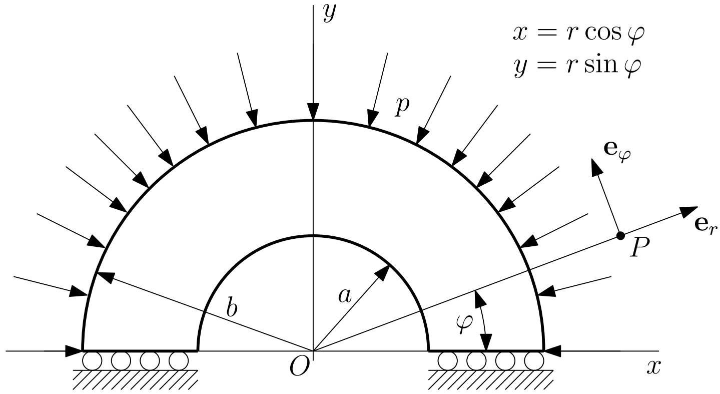

Figure 1. Thin elastic disc with uniform radial load

where $\mathbf{e}_r$ and $\mathbf{e}_\varphi$ are the unit vectors of the cylindrical coordinate system $Or\varphi$ (see Figure 1). The expressions of the strains are as follows [1–5]

$$\varepsilon_r = \frac{\partial u}{\partial r}, \qquad \varepsilon_\varphi = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \varphi}, \tag{1.7}$$

$$\gamma_{r\varphi} = \frac{1}{r}\frac{\partial u}{\partial \varphi} + \frac{\partial v}{\partial r} - \frac{v}{r}. \tag{1.8}$$

Based on the strain-displacement relationships [1–5] the following equations are valid

$$\varepsilon_r = \frac{\sigma_r}{E} - \nu\frac{\sigma_\varphi}{E}, \qquad \varepsilon_\varphi = -\nu\frac{\sigma_r}{E} + \frac{\sigma_\varphi}{E}, \tag{1.9}$$

$$\gamma_{r\varphi} = \frac{2(1+\nu)}{E}\tau_{r\varphi}. \tag{1.10}$$

In equations (1.9) and (1.10) σ_r and σ_φ are the normal stresses, $\tau_{r\varphi}$ denotes the shearing stress, E represents the modulus of elasticity and ν means the Poisson number. For this problem the equations of mechanical equilibrium are

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\varphi}}{\partial \varphi} + \frac{\sigma_r - \sigma_\varphi}{r} = 0, \qquad (r,\varphi) \in A, \tag{1.11}$$

$$\frac{\partial \tau_{r\varphi}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\varphi}{\partial \varphi} + \frac{2}{r}\tau_{r\varphi} = 0, \qquad (r,\varphi) \in A. \tag{1.12}$$

The following boundary conditions are prescribed in this problem:

$$\tau_{r\varphi} = 0,\ v = 0 \text{ on } \partial A_1, \tag{1.13}$$

$$\sigma_r = -p = \text{constant},\ \tau_{r\varphi} = 0 \text{ on } \partial A_2, \tag{1.14}$$

$$\tau_{r\varphi} = 0,\ v = 0 \text{ on } \partial A_3, \tag{1.15}$$

$$\tau_{r\varphi} = 0,\ \sigma_r = 0 \text{ on } \partial A_4. \tag{1.16}$$

The solution to the boundary value problem formulated by equations (1.7–1.14) will be solved under the conditions

$$u = u(r), \qquad v(r,\varphi) = 0, \qquad (r,\varphi) \in A \cup \partial A. \tag{1.17}$$

2. Plane stress solution

From equation (1.17) it follows that the equations of mechanical equilibrium are reduced to one equation, which is

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\sigma_r\right) - \sigma_\varphi = 0, \qquad (r,\varphi) \in A. \tag{2.1}$$

The general solution of stress equilibrium equation (2.1) in terms of stress function $F = F(r)$ can be represented for a thin elastic disc as

$$\sigma_r(r) = \frac{1}{t}\frac{F(r)}{r}, \qquad \sigma_\varphi(r) = \frac{1}{t}\frac{\mathrm{d}F(r)}{\mathrm{d}r}, \qquad (r,\varphi) \in A \cup \partial A, \tag{2.2}$$

where t is the thickness of the elastic disc. From equations (1.7) and (1.9) it follows that

$$Et\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{F}{r} - \nu\frac{\mathrm{d}F}{\mathrm{d}r}, \tag{2.3}$$

$$Et\frac{u}{r} = -\nu\frac{F}{r} + \frac{\mathrm{d}F}{\mathrm{d}r}. \tag{2.4}$$

The combination of equation (2.3) with (2.4) gives an ordinary second order differential equation for $F = F(r)$

$$\frac{\mathrm{d}^2F}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}F}{\mathrm{d}r} - \frac{F}{r^2} = 0, \qquad a < r < b. \tag{2.5}$$

According to the traction boundary conditions (1.14), (1.16) $F = F(r)$ satisfies the following boundary conditions

$$F(a) = 0, \qquad F(b) = -pbt. \tag{2.6}$$

The solution to the boundary value problem formulated by equations (2.5), (2.6) is

$$F(r) = \frac{b^2tp}{b^2-a^2}\left(-r + \frac{a^2}{r}\right), \qquad a \le r \le b. \tag{2.7}$$

The expressions of normal stresses σ_r and σ_φ can be represented as

$$\sigma_r(r) = \frac{b^2p}{b^2-a^2}\left(-1 + \frac{a^2}{r^2}\right), \qquad a \le r \le b, \tag{2.8}$$

$$\sigma_\varphi(r) = -\frac{b^2p}{b^2-a^2}\left(1 + \frac{a^2}{r^2}\right), \qquad a \le r \le b. \tag{2.9}$$

Based on equations (1.7), $(1.9)_1$ and (1.17) it is evident that

$$u(r) = \frac{r}{E}\left(\sigma_\varphi - \nu\sigma_r\right) \tag{2.10}$$

from which the following formula can be obtained for the radial displacement

$$u(r) = \frac{b^2p}{E(b^2-a^2)}\left[(\nu-1)r - (1+\nu)\frac{a^2}{r}\right], \qquad 0 \le r \le b. \tag{2.11}$$

3. Strength of material solution

The formulation of the strength of material solution is based on the paper [5], which uses the displacement field

$$\mathbf{u}(r,\varphi) = U(\varphi)\mathbf{e}_r + \left(r\phi(\varphi) + \frac{\mathrm{d}U}{\mathrm{d}\varphi} \right) \mathbf{e}_\varphi. \tag{3.1}$$

The corresponding strain field as a function of the displacement field given by equation (3.1) is

$$\varepsilon_\varphi = \frac{W(\varphi)}{r} + \frac{\mathrm{d}\phi}{\mathrm{d}\varphi}, \qquad W(\varphi) = \frac{\mathrm{d}^2 U}{\mathrm{d}\varphi^2} + U(\varphi). \tag{3.2}$$

Figure 2 shows the strength of material model according to paper [5]. The resultants of the tractions acting on the boundary surface segments ∂A_1 and ∂A_3 are F_1 and F_3. The moments of traction acting on the boundary surface segments ∂A_1 and ∂A_3 are M_1 and M_3 and we have $M_1 = M_3 = M_0$. The value of M_0 is obtained from the condition

$$\phi(\varphi) = 0, \qquad 0 \le \varphi \le \pi \tag{3.3}$$

according to the results presented in Section 2 of this paper. Application of the Hooke law gives

$$\sigma_\varphi(r,\varphi) = E\left(\frac{W(\varphi)}{r} + \frac{\mathrm{d}\phi}{\mathrm{d}\varphi} \right). \tag{3.4}$$

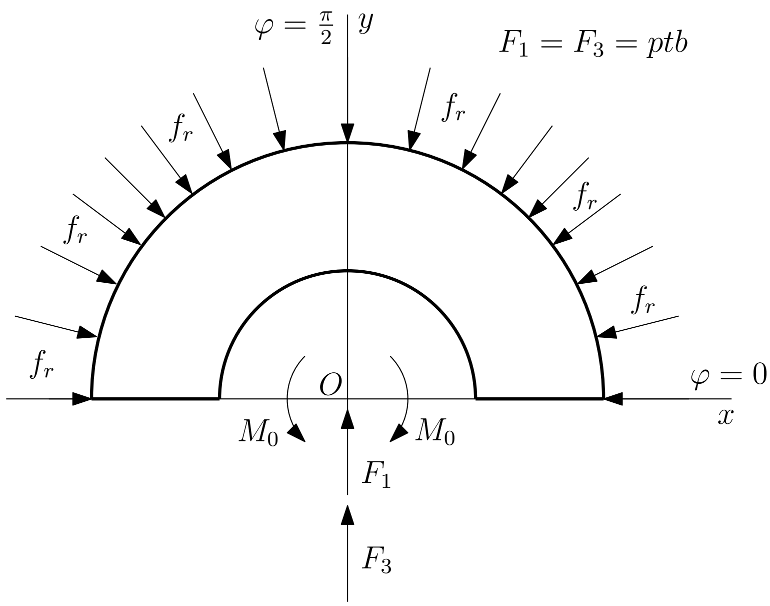

Figure 2. Strength of material model.

The normal force and bending moment on an arbitrary cross section can be calculated as

$$N=\int_A \sigma_\varphi \mathrm{d}A, \qquad M=\int_A r\sigma_\varphi \mathrm{d}A. \tag{3.5}$$

Detailed forms of expressions of $N=N(\varphi)$ and $M(\varphi)$ are as follows:

$$N(\varphi)=Et\left[W(\varphi)\ln\frac{b}{a}+(b-a)\frac{\mathrm{d}\phi}{\mathrm{d}\varphi}\right], \tag{3.6}$$

$$M(\varphi)=Et\left[W(\varphi)(b-a)+c(b-a)\frac{\mathrm{d}\phi}{\mathrm{d}\varphi}\right], \tag{3.7}$$

where

$$c=0.5(a+b). \tag{3.8}$$

In the present problem the solution of equilibrium equation [5]

$$\frac{\mathrm{d}^2N}{\mathrm{d}\varphi^2}+N-f_r=0 \tag{3.9}$$

is

$$f_r=-btp=\text{constant}, \qquad 0\le\varphi\le\pi. \tag{3.10}$$

The shear force $S=S(\varphi)$ vanishes since

$$S(\varphi)=-\frac{\mathrm{d}N}{\mathrm{d}\varphi}=0, \tag{3.11}$$

and from the moment equilibrium equation it follows that

$$M(\varphi)=M=\text{constant}, \qquad 0\le\varphi\le\pi. \tag{3.12}$$

If $\phi=0$ $(0\le\varphi\le\pi)$ then

$$M=-t\frac{(b-a)b}{\ln\frac{b}{a}}p, \tag{3.13}$$

$$W=\frac{\mathrm{d}^2U}{\mathrm{d}\varphi^2}+U=-\frac{bp}{E\ln\frac{b}{a}}. \tag{3.14}$$

Substitution of equation (3.14) into equation (3.4) gives

$$\sigma_\varphi(r)=-\frac{b}{r\ln\frac{b}{a}}p, \qquad a\le r\le b. \tag{3.15}$$

From the stress equilibrium equation

$$\frac{\mathrm{d}}{\mathrm{d}r}(r\sigma_r)=\sigma_\varphi \tag{3.16}$$

it follows that

$$r\sigma_r(r)-a\sigma_r(a)=-\frac{bp}{\ln\frac{b}{a}}\ln\frac{r}{a}, \qquad a\le r\le b, \tag{3.17}$$

that is

$$\sigma_r(r)=-\frac{bp}{r}\frac{\ln\frac{r}{a}}{\ln\frac{b}{a}}, \qquad a\le r\le b. \tag{3.18}$$

Thus, the normal stress σ_r given by equation (3.18) satisfies the stress boundary conditions

$$\sigma_r(a) = 0, \qquad \sigma_r(b) = -p. \tag{3.19}$$

Integration of equation (3.14) provides the radial displacement

$$U(\varphi) = -\frac{bp}{E \ln \frac{b}{a}} + \alpha \cos\varphi + \beta \sin\varphi, \tag{3.20}$$

where α and β are the constants of integration and

$$V(\varphi) = \frac{\mathrm{d}U}{\mathrm{d}\varphi} = -\alpha \sin\varphi + \beta \cos\varphi. \tag{3.21}$$

By means of the boundary conditions

$$V(0) = V(\pi) = 0, \qquad V\left(\frac{\pi}{2}\right) = 0 \tag{3.22}$$

it is easy to prove that

$$\alpha = \beta = 0, \tag{3.23}$$

so the radial displacement has the form

$$U(\varphi) = -\frac{bp}{E \ln \frac{b}{a}} = \text{constant}, \qquad 0 \le \varphi \le 2\pi. \tag{3.24}$$

4. Determination of Von Mises stress

In the present problem the equivalent Von Mises stress is obtained from the formula

$$\sigma(r) = \sqrt{\sigma_r^2(r) + \sigma_\varphi^2(r) - \sigma_r(r)\sigma_\varphi(r)}\,, \tag{4.1}$$

which yields the following result for the plane stress model

$$\sigma_1(r) = \frac{b^2 p}{(b^2 - a^2)\, r^2} \sqrt{r^4 + 3a^4} \tag{4.2}$$

and for the strength of material model

$$\sigma_2(r) = \frac{bp}{r \ln \frac{b}{a}} \sqrt{1 - \ln\frac{r}{a} + \left(\ln\frac{r}{a}\right)^2}\,. \tag{4.3}$$

5. Comparison of the solutions

In the following, the effect of the geometric parameters on the stresses is examined. First of all the radial normal stresses are considered. The radial normal stress is obtained from plane stress solution σ_r^{ps} [see equation (2.8)] and it can be reformulated in the following manner:

$$\sigma_r^{\mathrm{ps}} = \frac{pb^2}{b^2 - a^2}\left(-1 + \frac{a^2}{r^2}\right) = p\frac{1}{1 - \left(\frac{a}{b}\right)^2}\left[-1 + \left(\frac{a}{r}\right)^2\right]. \tag{5.1}$$

Similarly, the radial normal stress derived from the strength of material solution σ_r^{sm} [see equation (3.18)] can be written in the form

$$\sigma_r^{\mathrm{sm}} = -\frac{bp}{r}\frac{\ln\frac{r}{a}}{\ln\frac{b}{a}} = -p\frac{b}{a}\frac{a}{r}\frac{\ln\frac{r}{a}}{\ln\frac{b}{a}}. \tag{5.2}$$

New variables are introduced:

$$\lambda = \frac{a}{r}, \qquad \psi = \frac{a}{b}. \tag{5.3}$$

Since $a \le r \le b$ it is easy to prove that

$$0 < \psi < 1, \qquad \psi \le \lambda \le 1. \tag{5.4}$$

Substitution of equations (5.3) into equations (5.1) and (5.2) yields

$$\sigma_r^{\mathrm{ps}} = p\frac{1}{1-\psi^2}\left[-1+\lambda^2\right], \tag{5.5}$$

$$\sigma_r^{\mathrm{sm}} = -p\frac{\lambda}{\psi}\frac{\ln\frac{1}{\lambda}}{\ln\frac{1}{\psi}}. \tag{5.6}$$

Let $\Delta_r = \Delta_r(\lambda,\psi)$ denote the dimensionless difference of the radial normal stresses

$$\Delta_r(\lambda,\psi) = \frac{\sigma_r^{\mathrm{ps}} - \sigma_r^{\mathrm{sm}}}{p} = \frac{1}{1-\psi^2}\left[-1+\lambda^2\right] + \frac{\lambda}{\psi}\frac{\ln\frac{1}{\lambda}}{\ln\frac{1}{\psi}}. \tag{5.7}$$

Figure 3 illustrates the dimensionless difference function $\Delta_r(\lambda,\psi)$.

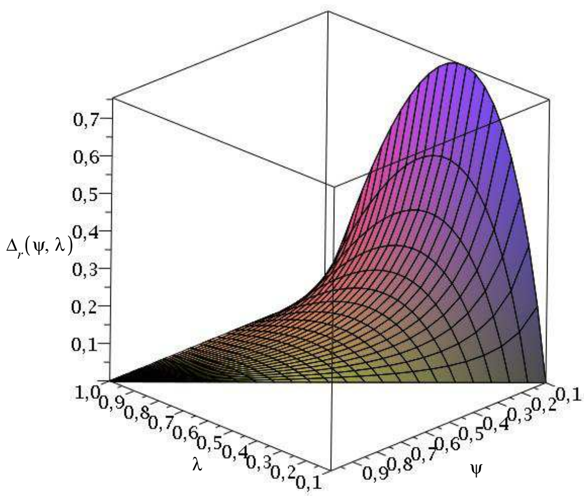

Figure 3. The dimensionless difference function $\Delta_r = \Delta_r(\lambda,\psi)$

The tangential normal stresses can be modified similarly. In the case of the plane stress solution, according to equation (2.8) one can write

$$\sigma_\varphi^{\text{ps}} = -\frac{b^2 p}{b^2 - a^2}\left(1 + \frac{a^2}{r^2}\right) = -p\frac{1}{1 - \left(\frac{a}{b}\right)^2}\left[1 + \left(\frac{a}{r}\right)^2\right]. \tag{5.8}$$

The modified form of the tangential normal stress in connection with the strength of material solution from equation (3.15) is as follows

$$\sigma_\varphi^{\text{sm}} = -\frac{b}{r \ln \frac{b}{a}} p = -p\frac{b}{a}\frac{a}{r}\frac{1}{\ln \frac{b}{a}}. \tag{5.9}$$

Substitution of equations (5.3) into equations (5.8) and (5.9) provides

$$\sigma_\varphi^{\text{ps}} = -p\frac{1}{1 - \psi^2}\left(1 + \lambda^2\right), \tag{5.10}$$

$$\sigma_\varphi^{\text{sm}} = -p\frac{\lambda}{\psi}\frac{1}{\ln \frac{1}{\psi}}. \tag{5.11}$$

Another dimensionless difference function denoted by $\Delta_\varphi = \Delta_\varphi(\lambda, \psi)$ can be established according to tangential normal stresses (5.10) and (5.11)

$$\Delta_\varphi(\lambda, \psi) = \frac{\sigma_\varphi^{\text{ps}} - \sigma_\varphi^{\text{sm}}}{p} = -\frac{1}{1 - \psi^2}\left(1 + \lambda^2\right) + \frac{\lambda}{\psi}\frac{1}{\ln \frac{1}{\psi}}. \tag{5.12}$$

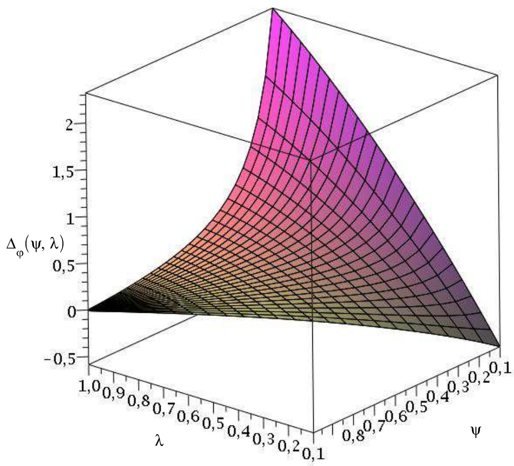

Figure 4. The dimensionless difference function $\Delta_\varphi = \Delta_\varphi(\lambda, \psi)$

Figure 4 shows the $\Delta_\varphi = \Delta_\varphi(\lambda, \psi)$ function. Figures 3 and 4 represent that the differences between the two solutions converge to zero when $\lambda, \psi \to 1$, which means that the outer radius b of the disc converges to the inner radius a (Figure 1). In that case the problem actually becomes a curved beam problem and then the two solutions are in good congruence. If $\psi \to 0$, namely the parameter b is significantly higher than a (so the disc is wide), then the difference between the two solutions increases.

6. Numerical examples

6.1. **Narrow disk.** The following data are used in the first numerical example: $a = 0.1$ m, $b = 0.2$ m, $E = 2\times 10^{11}$ Pa, $\nu = 0.3$, $p = 25\times 10^6$. According to the parameters $\psi = a/b = 0.5$ in this case. Investigating Figures 3 and 4, the example is close to a beam problem as the dimensionless difference functions (5.7) and (5.12) provide relatively low discrepancy between the two solutions. A plane stress FEM analysis has been also made to check and compare the results. In Figures 5 and 6 the plots of σ_r and σ_φ are shown as functions of r. The graphs of Von Mises stresses as a function of r are presented in Figure 7. Figure 8 represents the radial displacement functions as a function of r. The values of the radial displacement for $r = a$, $r = b$, $r = 0.5(a + b)$ are listed below

$$u(a) = -0.000033334 \text{ m}, \qquad u(b) = -0.0000341666 \text{ m},$$

$$u\left(\frac{a+b}{2}\right) = -0.0000319444 \text{ m}, \qquad U = -0.000036067 \text{ m}.$$

It can be clearly seen that the plane stress solution and the FEM solution (plane stress model too) produce practically the same results for all the stress and displacement functions. In this case the strength of material solution does not differ significantly from the plane stress results, either.

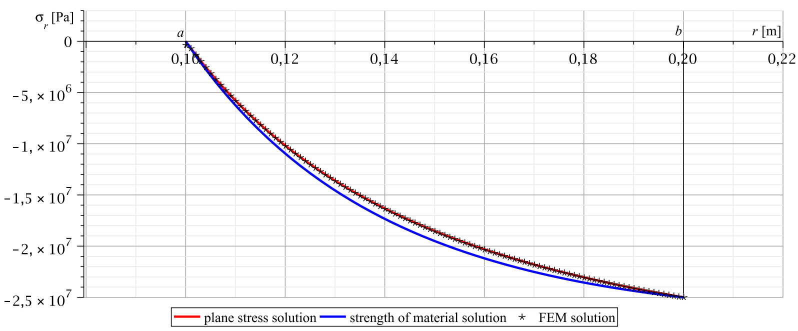

Figure 5. The plots of the radial normal stress functions $\sigma_r(r)$ (narrow disk)

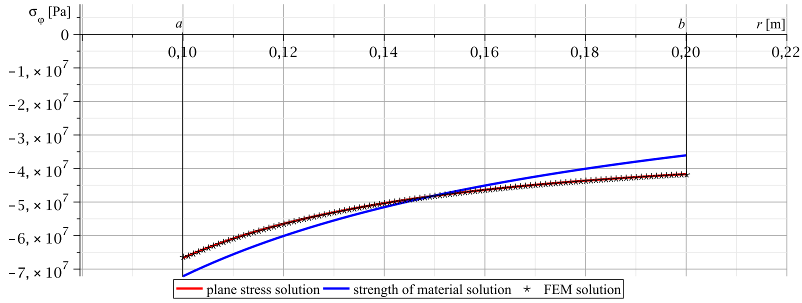

Figure 6. The plots of the tangential normal stress functions $\sigma_\varphi(r)$ (narrow disk)

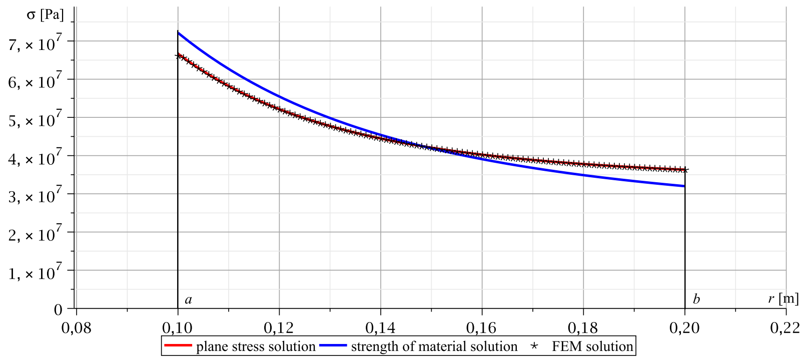

Figure 7. The plots of the Von Mises stress functions $\sigma(r)$ (narrow disk)

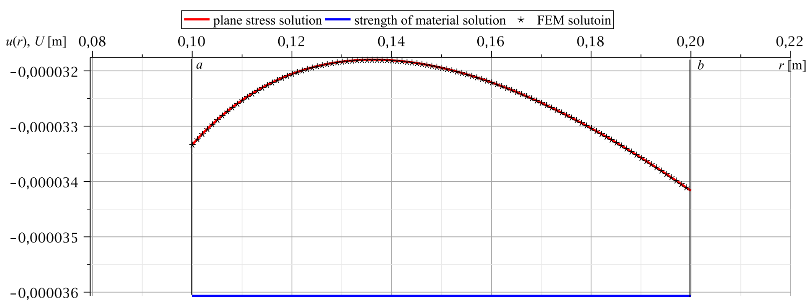

Figure 8. The plots of the radial displacement functions $u(r)$, U (narrow disk)

6.2. **Wide disk.** In this example a wide disk is analysed. The data are the same as in Example 6.1 with one exception. The outer radius of the disk $b = 1$ m. The ratio of the geometrical parameters $a/b = \psi = 0.1$, which means a much higher difference between the two analytical solutions according to dimensionless difference functions (5.7) and (5.12) (see Figures 3 and 4). A plane stress FEM analysis was carried out for this wide disk, as well. In Figure 9 the radial stress functions are shown as a function of r. The tangential stress functions as a function of r can be seen in Figure 10. Figure 11 represents the Von Mises stress functions in terms of r. The displacement functions in terms of r are also given in Figure 12. The values of the radial displacement for $r = a$, $r = b$, $r = 0.5(a + b)$ are listed below:

$$u(a) = -0.000025253 \text{ m}, \qquad u(b) = -0.000090025 \text{ m},$$

$$u\left(\frac{a+b}{2}\right) = -0.000051596 \text{ m}, \qquad U = -0.000054287 \text{ m}.$$

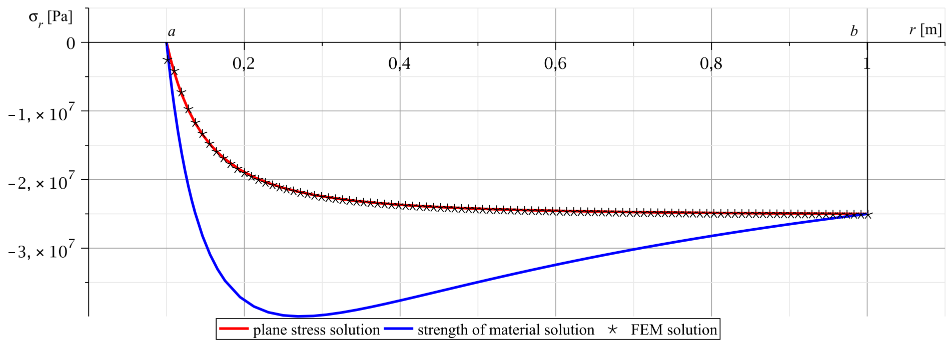

Figure 9. The plots of the radial normal stress functions $\sigma_r(r)$ (wide disk)

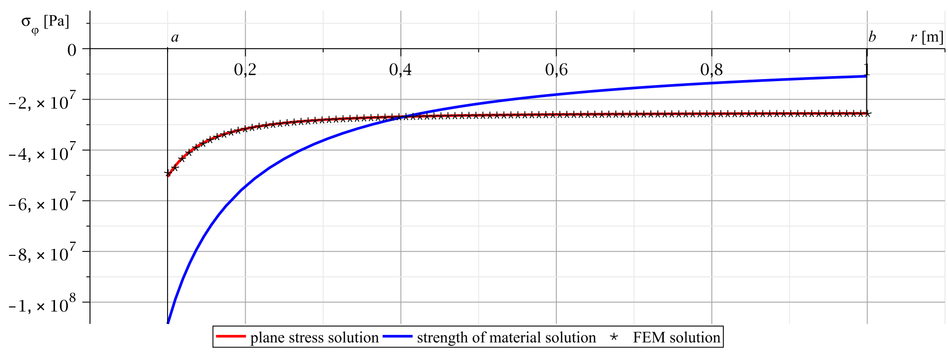

Figure 10. The plots of the tangential normal stress functions $\sigma_\varphi(r)$ (wide disk)

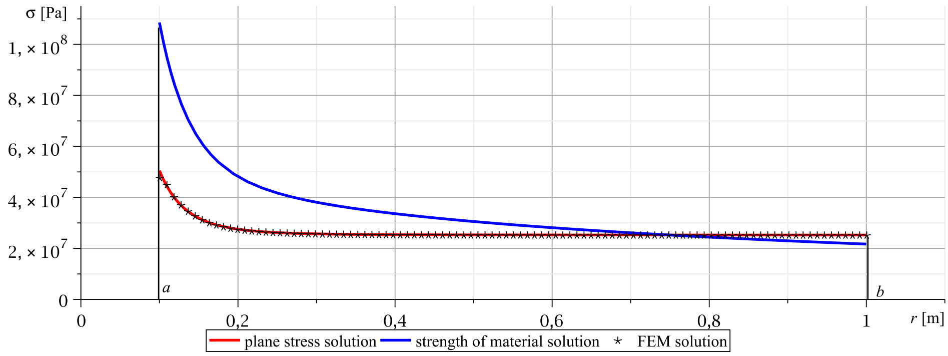

Figure 11. The plots of the Von Mises stress functions $\sigma(r)$ (wide disk)

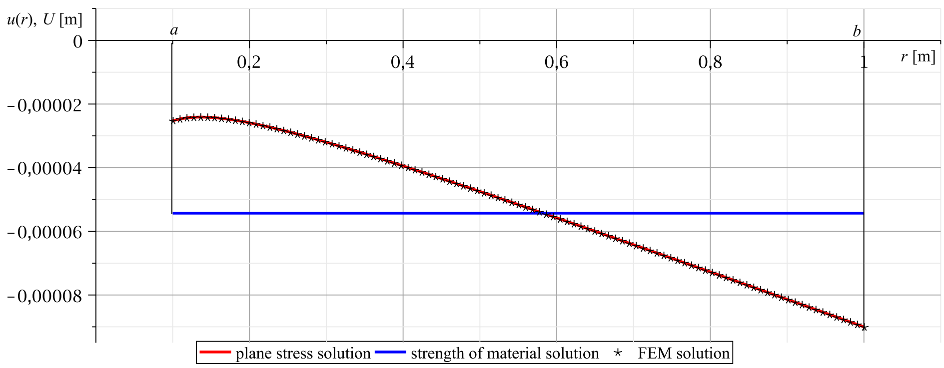

Figure 12. The plots of the radial displacement functions $u(r)$, U (wide disk)

It can be concluded that the plane stress solution and the FEM solution (plane stress model too) yield the same results for all of the stress and displacement functions as in the previous example 6.1. In this case the strength of material solution significantly differs from the plane stress results, as was expected.

7. Conclusions

The investigation of the state of stresses of a thin elastic disc is presented by applying two different mechanical models. The first model uses the governing equation of the plane stress deformation. The second model is a strength of material model. The results derived from the two models are in good agreement when the thin disk problem converges to a curved beam problem according to the geometrical parameters. When the disk becomes wider the results of the models diverge. Numerical examples illustrate the application of the derived formulae.

The results of the calculations illustrate that the radial and tangential normal stresses calculated with the two different models differ slightly when the thin disk can be considered as a curved beam according to the geometrical parameters. The same remark applies to radial displacements. The examples also represent the divergence of the stresses and the displacements derived from the two models in the case of a wide disk. The examples were also investigated with FEM analysis to check the results of the two models.The plane stress FEM analysis yields practically the same results as the analytical plane stress model in all examples investigated.

References

1. Sokolnikoff, I. S. *Mathematical Theory of Elasticity.* McGraw-Hill, New York, 1956.
2. Timoshenko, S. and Goodier, J. N. *Theory of Elasticity.* McGraw-Hill, New York, 1970.
3. Southwell, R. V. *An Introduction to the Theory of Elasticity.* Oxford University Press, Oxford, 1936.
4. Love, A. E. H. *A Treatise on Mathematical Theory of Elasticity.* University Press, Cambridge, 1952.
5. Ecsedi, I. and Dluhi, K. "A linear model for the static and dynamic analysis of non-homogeneous curved beams." *Applied Mathematical Modelling,* **29**(12), (2005), pp. 1211–1231. doi: 10.1016/j.apm.2005.03.006.